A Foundation for Mathematical Kinds

Matthew Trager

Introduction

Natural kinds are thought to be privileged divisions of reality independent of human classification. They are often praised for things such as permitting fruitful inductive inferences in the natural sciences. So far, however, there have been few attempts to apply natural kindhood to mathematics. In this essay, I will attempt to provide an account of natural or real mathematical kindhood of the natural numbers. First, I will paint a realist picture of mathematics which can accommodate natural kinds. Then I will then defend an essentialist account of natural numbers as a real mathematical kind. Afterwards, I will explore some paths forward to apply natural kinds in the philosophy of mathematics.

A Picture of Mathematics that Accommodates Kinds

As often discussed in the philosophy of science, there are certain criteria thought important or necessary for a kind to be considered a natural one. Two important criteria are:[1]

1) *Naturalness or Realness*: Natural kinds are thought to be metaphysically privileged divisions of reality independent of human classification. To use the old Platonic saying, they "carve nature at its joints".
2) *Hierarchy*: Natural kinds form a natural hierarchy. Lower-level kinds build into higher level kinds, and there might exist on the hierarchy *sub kinds* of another kind.

[1] Many from (Tobin and Bird, 2022)

In the case of criterion (1), natural kinds are often thought to be real, privileged divisions of reality because of some properties (typically thought to be intrinsic properties) which all members of that kind share. Without real properties, there is no non-arbitrary way to classify kinds in a way that reflects the structure of reality. So, it seems in order to establish natural mathematical kinds, there must similarly be some real properties or divisions which permit real classifications. Criterion (2) stresses the importance of a hierarchy of kind classifications, as it suggests there is a best classification scheme of reality (Khalidi, 1993).

Any legitimate attempt to create an account of natural kinds in mathematics must fulfill criteria (1) and (2). Therefore, I will proceed to spell out a realist mathematics which I hope is uncontroversial to those in favor of mathematical realism. The aim is to provide a foundation for an account of a natural mathematical kind.

Realism

The natural kind realist asserts that natural kinds reflect the privileged, metaphysical distinction in the natural world. It is natural just in case that it is a classification of this reality independent of human classifications. In this sense, the natural is the real. And it would seem that anti-realist views of mathematics, such as constructivism and formalism, immediately discount real natural kinds in mathematics. Formalist views of mathematics suggest no real mathematical entities; math is merely the manipulation of symbols. Constructivist views of mathematics conflict with the criterion that kinds must exist independently of human classifications. If mathematical entities are human constructions, then they cannot be real. We are left with some form of mathematical realism—whether this be platonism, logicism, or some other realist philosophy—to fulfill the realist criterion. So, in order to give an account of natural kinds in mathematics, I will assume some kind of "agnostic" mathematical realism. Thus, if natural kinds in mathematics exists, they can be understood as 'real' or 'natural' kinds as concerning actual, abstract mathematical objects and properties; analogous to the 'real' or 'natural' kinds in the natural sciences concerned with physical objects and properties. With this general realist framework, we can proceed to argue for natural kinds in mathematics, or as I will call them from now on, 'real mathematical kinds.[2]'

Hierarchy

However, we must make an addition to our mathematical realism—one that establishes a hierarchy for real mathematical kinds. While a hierarchy is becoming more controversial in natural kind literature,[3] some notion of hierarchy is important to maintain. Lower-level kinds are often thought to explain and constitute higher-level kinds—like hydrogen by its protons, neutrons, and electrons. Protons, neutrons, and electrons are themselves kinds, and made up by other proposed kinds, such as quarks. This is a hierarchy from simple to more complex phenomena. Similarly, any overlapping kinds are thought to be sub kinds of the other (Bird and Tobin, 2022). 'Hydrogen' is of the kind atom, but that is because 'hydrogen' itself is a sub kind of atom, as is 'helium'. So, while there is controversy about the exact structure of the hierarchy, it is important to flesh out some organizational schema. Natural kinds ultimately aim to organize reality, so the rules of their organization must be established.

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[2] "Natural kinds in mathematics" has been shown to be synonymous with "real mathematical kinds". This is how Corfield (Corfield, 2004) refers to natural kinds in mathematics as well. The renaming is also done for the sake of readability, as statements like "the natural numbers are a natural kind" can get confusing.
[3] Part of this controversy is due to things like "cross-cutting kinds" (Tobin, 2010).

So, what might a hierarchical structure of mathematics look like? Here I will appeal to the work of mathematician Bernard Bolzano and his ideas about grounding in mathematics. This grounding produces a sort of mathematical hierarchy. Establishing a mathematical truth by means of proof, says Bolzano, cannot merely be an epistemic justification by deductive methods. They must be based on good method, but also upon the proper 'grounds' (Russ, 1980). These grounds are themselves metaphysical, meant to get to the "why" questions of mathematics. To Bolzano, grounding is the relation between mathematical propositions—the *very structure* by which mathematical propositions relate (LaPointe, 2008). The structure of these relations itself has explanatory value and underpins mathematical truths, taking the form of "q grounds p". As each proposition is itself grounded by a lower-level concept, real justification starts with primitively true concepts and builds up, providing the proper grounds for a new mathematical proposition (LaPointe, 2008). In order to create a proper proof of a mathematical proposition, the mathematician must evoke a notion of 'formal grounding,' where they use both the grounds of a mathematical truth and a formal deduction (LaPointe, 2008).

To elaborate, consider an example where proof goes wrong: using a geometric proposition in a proof of mathematical analysis. To Bolzano, geometric truths are not simple and are of a higher-order than propositions of analysis. Thus, they cannot be used to ground truths of analysis (Russ, 1980). Instead, we must invoke primitively true concepts as the grounds for any mathematical proof and build from the bottom up.

We can now see how Bolzano's metaphysical hierarchy and reductionist attitude produces fertile ground for real mathematical kinds. The major takeaways here are 1) that there is some reason "why" for mathematical truth, ultimately understood as a real mathematical *structure*, 2) that these real structures build upon each so that we can 3) explain higher-level mathematical structures in terms of lower-level ones. Furthermore, invoking Bolzano's philosophy isn't particularly controversial for the mathematical realist. Analytic method is commonly accepted by mathematicians and we can furthermore treat his notion of structure *agnostically*. A mathematical structure could be a platonic form or a logical object.[4] So overall, this presents a picture of real mathematics with promise for real mathematical kinds.

An Essentialist Account of the Natural Numbers

We now have a realist mathematics which posits real mathematical structure and a hierarchical form. With an overall picture in mind, we can begin to identify kinds in mathematics. The paradigm case for a real mathematical kind will be *natural number.* In fact, I suggest that if the natural numbers cannot be established as a real mathematical kind, then any hopes of establishing real mathematical kinds should be abandoned. I will begin by outlining the various accounts of kindhood and ultimately argue that an essentialist account is the best fit for mathematics, and therefore *natural number*. Then I will outline two different essentialist accounts of *natural number* and weigh the pros and cons of each. I will ultimately suggest that we define *natural number* in terms of an extrinsic property.

To establish a kind, we must have an account of what it is to be of that kind. The two most popular definitions of a kind are Homeostatic Property Cluster accounts and essentialist accounts. In the Homeostatic Property Cluster (HPC) view, kinds are clusters of properties held together by a shared homeostatic mechanism (Boyd, 1999). A paradigm case for HPC kinds are something like tigers. Tigers generally have properties like *being stripped* or *being orange* held together by a homeostatic biological mechanism. On the other hand, the essentialist account

[4] Notably, Bolzano was anti-Kantian (LaPointe, 2008). This could provide an issue for a Kantian conception of real mathematical kinds.

states that to be a member of the kind X, there is some essential property Y (or properties Y1, Y2, etc.) to be that kind. It is commonly thought that the essential property is both necessary and sufficient to that kind. For example, take the kind *hydrogen*. Its essential property would be *having one proton*. If a thing had zero or two protons, it cannot be a hydrogen atom.

There is the question about which account of kindhood best fits with mathematics. I propose, and I think it is obvious, that the essentialist account is the way to go. For starters, mathematics is often thought to be a science of necessary truths, so it is intuitive that mathematical kinds have essential properties. The HPC account does not appeal to essential properties, and it was created to account of kinds which do not fit essentialist molds. Scientists and philosophers might consider *tigers* to be a real kind, despite the fact there is no clear essential property of *tiger*. Generally, tigers have stripes and four legs, but this is not true for all tigers. *Tiger* is a cluster of properties, none being necessary or sufficient for *tigerhood*. The identification of the homeostatic mechanism or close proximity of an individual's properties to a kind is sufficient to identify it as a member of a kind. But this modification (perhaps even a concession to natural kind anti-realists) is not required for establishing the kindhood of natural number. So, a promising start in our picture of mathematics is an *essentialist* account. Spelled out more clearly:

> *Essentialist Account*: For a thing X to be a member of kind K, there is some property Y (or properties Y1, Y2, etc.…) which are essential to being kind K. Property Y (or properties Y1, Y2, etc.) are necessary and sufficient to be of kind K.

In order to identify the natural numbers as a real mathematical kind, we must identify a real essential property which each member of the natural numbers (0, 1, 2, 3, etc.) has. The obvious next step is appealing to something like Bolzano's *structure* or the *structure of natural* *number*—the real thing which grounds the existence and truth of a mathematical entity. Indeed, appealing to a structure to define a natural or real kind is commonplace in the philosophy of science. For example, H2O is the essential (micro)structure of water, and establishes water as a natural kind (Bird and Tobin, 2022). Similarly, the structure C6H12O6 is thought essential to glucose. So, an essentialist account states that a number *n* has an essential structure which makes it a member of *natural number*.

So, maybe we can leave it at that. A number *n* is a member of the real mathematical kind *natural number* insofar as the *structure* of natural number is an essential property of *n*. However, we shouldn't be so hasty. We may require a more nuanced definition of the natural numbers.

The essential properties of a natural kind are often thought to be intrinsic properties, as opposed to extrinsic ones. The paradigm cases of kinds, such as *hydrogen* and *water,* each have essential structures which are intrinsic to their nature. On this view, the *natural number structure* of a number *n* is an intrinsic property of that number.

But the view that there is some intrinsic property of each number might receive some pushback. For one, it is unclear about what an intrinsic property of an abstract entity is. It is rather intuitive for natural entities to have intrinsic properties which constitute that entity—a water molecule is made up of H2O, and *water* can be defined as being identical to H2O. But consider famous definitions of the natural numbers, notably Gottlob Frege's *Foundations of Arithmetic.* Frege defines an *ancestral relation* that holds between two elements *x* and *y*. Here, it is useful to think of a number being the parent of the following number (0 is the parent of 1, and so on). He then defines a predecessor relation where *x precedes y*, if for a concept F and an object z that falls under F, y is the number of the concept F and x is the number of objects falling F besides z. This defines a number *n* in terms of its predecessor—a natural number is 0, or 0 is an ancestor of *n* (Frege, 1959). In sort, this allows him to define a successor in terms of its predecessor, where *n* is ancestrally related to 0 through a successor relationship. Details aside, this ancestral relationship and successor relationship I suggest, gets at the *structure* of natural number. There is the relation between each individual number, and this ultimately constitutes the larger structure of natural number. But it seems that Frege does not outline an intrinsic essence, but rather an extrinsic one. Each number (0, 1, 2, etc.) does not share the essential property of 'the structure of natural number'. Each individual number is an entity which *constitutes* or participates in the overall structure of *natural number*. It is analogous to bricks constituting the structure of a house. The bricks make the house, but themselves are not the members of the kind house. So, a number could be thought to *constitute* the structure of natural number without reference to an intrinsic essence.

There are two competing essentialist definitions. The first appeals to an intrinsic essence, perhaps understood as a *structure*, which each natural number shares. 'Structure' is best thought of as an abstract universal which each natural number instantiates. It is an intrinsic 'natural number-ness', not a thing each natural number has an extrinsic relation to.

> *Intrinsic:* A number *n* is a member of the real mathematical kind *natural number* insofar as the *structure* of natural number is an essential property of *n*.

Or there is the extrinsic view, where *n* is a natural number due to its *relation* to overall the mathematical structure:

> *Extrinsic: n* is a member of the real mathematical kind *natural number* insofar as *n* has the essential property *constituent of the structure of the natural numbers*.

Case for Extrinsic Definition

I have shown that *natural number* can be considered a real mathematical kind, but there are two competing definitions: one that proposes an essential intrinsic property, and one that proposes an essential extrinsic property. Ultimately, I prefer an extrinsic definition.

I wish to put fears aside that the extrinsic definition of *natural numbers* is solely done due to pressures from logicism—a philosophy long rife with trouble since Russell's Paradox. The account is heavily informed by logicism. However, the account is compatible with any real structure of the natural numbers—not just Fregean logical objects. A platonist, to wit, would have no trouble appealing to some real structure. Frege's ancestral relation could then be thought to be simply "tapping into" the structure of natural numbers, in the same way that Peano's Axioms might also tap into or "get at" that real structure. Their elaboration on natural numbers makes the essential property of *natural number* something less mysterious than an abstract universal of *natural number-ness*.

However, intrinsic *natural number-ness*. Need not be so mysterious. Consider that any natural number n is followed by another natural number n+1. This might suggest that each individual natural number might have the property of *having a successor* or *having a predecessor*.[5] The essence could be something like "countability" or "discreteness", and these would be intrinsic essences.[6]

However, I believe the extrinsic account of natural numbers accommodates for an important fact about the kind *natural number*: it has *necessary* members. To explain, assume that there is a real kind *tiger* and that each individual tiger instantiates some essence of *tiger-ness*. But consider that if some tiger had never happened to be born, nothing about the essence of

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[5] This objection was suggested by my professor Janet Folina.
[6] This was suggested by my friend Amanda Jackson.

tiger-ness would have changed. Similarly, if a particular hydrogen atom was never produced from radioactive decay, nothing about the essence of hydrogen (it's one proton microstructure) would have changed. However, this isn't the case for the natural numbers. It is inconceivable that the natural numbers could not have included '6'. And if the natural numbers did not include '6', then the very notion of what it means to be a natural number would be different. So, the natural numbers have necessary members.

I propose that the extrinsic definition has explanatory value: it can explain why the natural numbers have necessary members. To elaborate, I will draw a comparison to clades in biology. Clades are thought to be an essentialist classification of organisms, one to replace talk about species (Osaka, 2002). A clade is defined and composed of an organism (an "ancestor") and all of its descendants (Bird and Tobin, 2022). This makes each member of the clade necessarily a member of the clade, and establishes the essential property of being a member of a clade by the lineage the member is a part of—an extrinsic property. Hence, if a particular member of a clade never happened to be born, the very meaning of that clade (say *tiger*) would be different. So, the clade has necessary members because the members themselves constitute the clade. This means that fact defining how natural number constitutes the very structure of natural number provides an explanation as to why the kind *natural number* has necessary members. An extrinsic property is not a problem, and it has explanatory power.

This explanatory power, I believe, provides a suitable reason why we should define *natural number* by its extrinsic relation to the overall structure of natural number. I will admit that this debate is not conclusive. For example, the fact that there are necessary members of natural number might be because of the brute fact that all mathematical truths are necessary ones.

The essentialist definition of *natural number* is still up for debate. But for now, I believe to established a sufficient account of the real mathematical kind *natural number*.

Further Implications

Before I conclude, there are further implications from my account of real mathematical kinds I would like to explore.

Prime Numbers and Laws of Nature

Now that we have established natural numbers as a real mathematical kind, we can give an account of higher-order mathematical kinds. An example of this is the kind prime number. We can perhaps say that:

> For *n* to be a member of the kind 'prime number', *n* is a natural number with the essential property of 'not divisible by any other natural number other than itself and 1'.

Here, we have defined what it is to be a prime number by defining a number in terms of the natural numbers. For example, *n* itself must be a natural number. Any prime number, then, is shown to be a *subset* of the kind 'natural number'. Here we can see our notion of hierarchy in practice: primes are subsets of natural number, and primes are furthermore defined in terms of the natural numbers.

However, you might notice we have not spelt out exactly what is meant by *divide*, *multiply*, *subtract*, etc. How do these relate to natural kindhood? While I will not cover it in detail here, notions like *divide* could be thought of similarly to the laws of nature (Corfield, 2003). It is often thought a property of natural kinds that they participate in the laws of nature

(Bird and Tobin, 2022), so this seems like a promising direction to explore the notion of real mathematical kinds.

An Epistemic Payoff?

Another important feature of natural kinds I have not yet touched on is their *epistemic role* in a scientific discipline. In particular, their ability to ground good inductive inferences. It has even been suggested that introducing natural kind talk to mathematics would allow for non-deductive inferences in mathematics (Baker, 2020). Non-deductive methods in mathematics are a controversial subject, and one I certainly cannot do justice in this essay. I will not make any claims about non-deductive methods here. However, this provides a strong motivation to adopt real mathematical kinds: an epistemic payoff.

There is an obvious payoff to definition mathematical reality by real mathematical kinds. For example, David Corfield argues that a kind hierarchy entails organizational epistemic value (Corfield, 2003). By establishing an effective hierarchy or classificatory matrix, a mathematician can place a new mathematical discovery in the context of the hierarchy. This allows mathematicians to make referential contact with a new mathematical entity, even though they don't get it "quite right" (Corfield, 2003). Drawing from Corfield's argument, say I'm walking through the forest and I see a monkey holding a banana. However, I mistake what I saw and claim to have seen an "ape holding a papaya". With an established hierarchy for organization, we can portray how far away from the mark my classification was. For example, if I said I saw a "bird with a gun" in the tree, my description would be farther from the mark than "ape holding a papaya". This is an immediate epistemic return gained by establishing a real mathematical kind hierarchy. Induction remains a tricky issue, one that deserves its own essay.[7]

Conclusion

In conclusion, I have provided an account of the natural numbers as a real mathematical kind. First, I established a picture of real mathematics that accounts for real mathematical kinds. Then I provided an essentialist account of the natural numbers by way of an extrinsic relation between a number and the overall structure of natural numbers. Finally, I briefly explored further implications and payoffs of adopting real mathematical kinds.

[7] The natural numbers do establish mathematical induction. This could be a further path of exploration.

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